Übungsstunde 4

Nachbesprechung Bonus

- $\forall x \forall y \forall z$ statt $\forall x, y, z$
- $\Leftrightarrow$ nicht in Formeln!

3.2 From Natural Language to a Formula ($\star$) ***(4 Points)***

Consider the universe $U = \mathbb{N} \setminus \{0\}$. Express each of the following statements with a formula in predicate logic, in which the only predicates appearing are $\texttt{divides}(x, y)$, $\texttt{equals}(x, y)$ and $\texttt{prime}(x)$ (instead of $\texttt{divides}(x, y)$ and $\texttt{equals}(x, y)$ you can write $x \mid y$ and $x = y$ accordingly). You can also use the symbols $+$ and $\cdot$ to denote the addition and multiplication functions, and you can use constants (e.g., $0, 1, \ldots$). You can also use $\longleftrightarrow$. No justification is required.

i) ($\star$) If a number divides two numbers, then it also divides their sum.

ii) ($\star$) The only divisors of a prime number are 1 and the number itself.

iii) ($\star$) 1 is the only natural number which has an inverse.

iv) ($\star$) A prime number divides the product of two natural numbers if and only if it divides at least one of them.

Nachbesprechung Bonus

3.8 Proof by Contradiction ($\star$) ***(4 Points)***

Let $n, m \in \mathbb{N}$ be arbitrary. We say "n divides m" and write $n \mid m$ if there exists a $k \in \mathbb{N}$ such that $k \cdot n = m$. Prove that the following statement is true, using a proof by contradiction:

$$n \mid m \text{ and } n \mid (m+1) \implies n = 1.$$

You are allowed to invoke the statement 3.2 **iii)** from above to justify one step.

You must use the same notation as in the lecture notes, i.e. precisely state what your statements S and T are, and justify each of your proof steps.

Definition 2.17. A *proof by contradiction* of a statement S proceeds in three steps:

1. Find a suitable mathematical statement T.
2. Prove that T is false.
3. Assume that S is false and prove (from this assumption) that T is true (a contradiction).

Mengenlehre

Einführung

- Uns schon bekannte Mengen sind z.B.: $\mathbb{N}, \mathbb{R}, \{0,1,2\}, \emptyset$
- In Mengen spielt Reihenfolge und Häufigkeit keine Rolle:
  $\{1,2\} = \{2,1\} = \{2,1,2\}$
- Mengen können Mengen enthalten:
  $\{1, \{1\}, \{5, \{3,2\}\}, \emptyset\}$

Element und Teilmenge

- $x \in A$: x ist ein **Element** der Menge A

  $2 \in \mathbb{N}, -1 \notin \{0,2,4\}, \{1,2\} \in \{\{1,2\}, 6\}$

- $X \subseteq A$: Die Menge X ist eine **Teilmenge** der Menge A

  $\{2,3\} \subseteq \{1,2,3,4,5\}, \{1,2\} \nsubseteq \{1, \{2\}\}$

  Für jede Menge A gilt: $\emptyset \subseteq A, A \subseteq A$

Aufgaben

Wir haben folgende Mengen:

$A = \{1,2\}, \qquad B = \{\{1\}, 2, \{1\}, \{\emptyset\}\}, \qquad C = \{1,1\}, \qquad D = \{\{1\}\}, \qquad E = \emptyset$

Welche Beziehungen gelten zwischen den Mengen?

Definitionen

- Vereinigung: $x \in A \cup B \Leftrightarrow x \in A \vee x \in B$
- Schnittmenge: $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$
- Differenz: $x \in A \backslash \mathrm{B} \Leftrightarrow x \in A \wedge \neg(x \in B)$

- Potenzmenge: $\mathcal{P}(A) = \{S \mid S \subseteq A\}$

  Für eine endliche Menge A gilt: $|\mathcal{P}(A)| = 2^{|A|}$
- Kartesisches Produkt: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

  Für endliche Mengen A und B gilt: $|A \times B| = |A| \cdot |B|$

Beispiele – Elemente und Kardinalität

- $\mathcal{P}(\{1,2,3\})$
- $\left(\{\emptyset\} \cup \left\{\{\emptyset\}\right\}\right) \setminus \{\emptyset\}$
- $\left\{2, \{4\}\right\} \times \{\{0\} \cup \{1\}, \{1,0\}\}$
- $\{0,1\} \times \{(0,1)\}$
- $P\left(\left\{\emptyset, \{\emptyset\}\right\}\right)$

Gleichheit von Mengen beweisen/widerlegen

Beweise oder widerlege:

Für alle Mengen A,B,C gilt: $(A\setminus \mathrm{B})\setminus \mathrm{C} = A\setminus(B\setminus \mathrm{C})$

Gleichheit von Mengen beweisen/widerlegen

Beweise oder widerlege:

Für alle Mengen A,B,C gilt: $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Definition 3.2. $A = B \overset{\text{def}}{\iff} \forall x\,(x \in A \leftrightarrow x \in B).$

Klausurfragen

Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. List all subsets S of A such that $\emptyset \in S$. *(2 Points)*

Show two sets A, B such that $A \cap B \in B$. *(1 Point)*

How many elements does the set $\{\{0,1\}, \{0\} \times \{1\}\} \times \{0\}$ have?

3.) Find sets A, B, and C such that $|A| = |B| = |C| = 2$, $|A \cap B| = |B \cap C| = |A \cap C| = 1$, and $A \cap B \cap C = \varnothing$.